Bayesian Modeling of Dependence with Copulas

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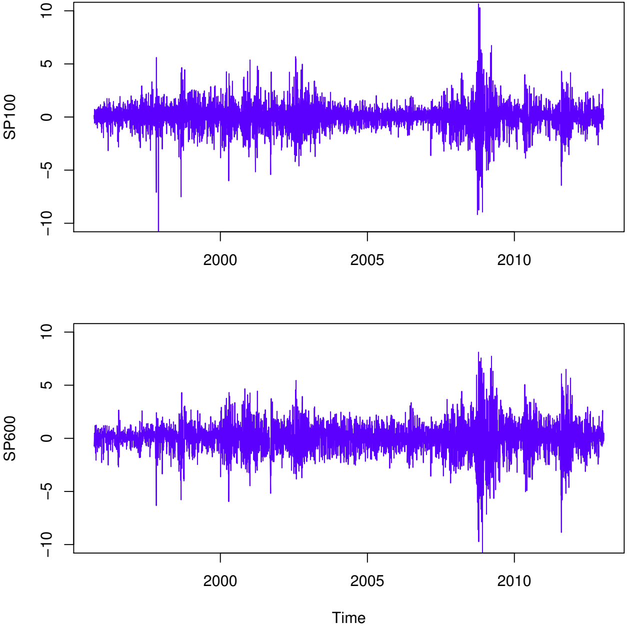
If I should ever have a tattoo, that would be

$$p(\theta|y) \propto p(y|\theta)p(\theta).$$

The stock market returns



Our interests

- We would like to construct a multivariate model that some margins are continuous but some are discrete.
- We would like to dynamically model the rank correlations:

$$\tau = 4 \int\int F(x_1, x_2) dF(x_1, x_2) - 1.$$

- As well as the tail dependences

$$\lambda_L = \lim_{u \to 0^+} \Pr(X_1 < F_1^{-1}(u) | X_2 < F_2^{-1}(u))$$

- We will see why we can not do it (easily) in the frequentist approach.

Preliminary

- The word "copula" means **linking**.
- **Sklar's theorem**

  Let H be a multi-dimensional distribution function with marginal distribution functions $F_1(x_1), ..., F_m(x_m)$. Then there exists a function C (**copula function**) such that

$$\begin{aligned} H(x_1, ..., x_m) =& C(F_1(x_1), ..., F_m(x_m)) \\ =& C\left(\int_{-\infty}^{x_1} f(z_1)dz_1, ..., \int_{-\infty}^{x_m} f(z_m)dz_m\right) = C(u_1, ..., u_m). \end{aligned}$$

  Furthermore, if $F_i(x_i)$ are continuous, then C is unique, and the derivative $c(u_1, ..., u_m) = \partial^m C(u_1, ..., u_m)/(\partial u_1 ... \partial u_m)$ is the **copula density**.

- The Joe-Clayton copula function

$$C(u, v, \theta, \delta) = 1 - \left[1 - \left\{\left(1 - \bar{u}^{\theta}\right)^{-\delta} + \left(1 - \bar{v}^{\theta}\right)^{-\delta} - 1\right\}^{-1/\delta}\right]^{1/\theta}$$

  where $\theta \geqslant 1$, $\delta > 0$, $\bar{u} = 1 - u$, $\bar{v} = 1 - v$.
- Our interests:
  - The rank correlation and tail dependence in the model.
  - The convenience for interpretation (knowing the underlying factors of dependences).
- We use the reparameterized copula $C(u, v, \lambda_L, \tau) = C(u, v, \theta, \delta)$.

* **Note!** any other copulas can be equally well used.

The covariate-contingent copula model

↪ The model

- **The marginal models**
  - In principle, any combination of univariate marginal models can be used.
  - In the continuous case, we use univariate model that each margin is from the student *t* distribution.
- **The log likelihood**

$$\begin{aligned}\log p(\{\beta, \mathfrak{I}\}|y, x) =& \text{constant} + \sum_{j=1}^{M} \log p(y_{.j}|\{\beta, \mathfrak{I}\}_j, x_j) \\ &+ \log \mathcal{L}_C(u|\{\beta, \mathfrak{I}\}_C, y, x) + \log p(\{\beta, \mathfrak{I}\})\end{aligned}$$

where all the parameters are connected with covariates via link function $\varphi(\cdot)$, (identity, log, logit, probit,...)

$$\begin{array}{lll} \text{Marginal features} & \mu = \varphi_{\beta_u}^{-1}(X_u \beta_u), & \sigma^2 = \varphi_{\beta_\sigma}^{-1}(X_\sigma \beta_\sigma), \ldots \\ \text{Copula features} & \lambda_L = \varphi_{\lambda}^{-1}((X_u, X_v)\beta_\lambda), & \tau = \varphi_{\tau}^{-1}((X_u, X_v)\beta_\tau). \end{array}$$

- The priors
  - The priors for the copula functions are easy to specify due to our reparameterization.
  - The priors for the marginal distributions are specified in their usual ways.
  - When variable selection is used, we assume there are no covariates in the link functions *a priori*.
- The posterior

$$p(\boldsymbol{\beta}|\mathbf{Y}) \propto \mathcal{L}(\mathbf{Y}|\beta) \times \prod_{i \in u, v, \tau, \lambda} p(\beta_i)$$

- The posterior inference is straightforward although the model is very complicated.

The beauty of Bayesian approach is not because of its complexity, but because of its simplicity.

- We update all the parameters jointly by using Metropolis-Hastings within Gibbs.
- The proposal density for each parameter vector β is a multivariate t-density with $\mathrm{df} > 2$,

$$\beta_p|\beta_c \sim \mathbf{MVT}\left[\hat{\beta},\ -\left(\frac{\partial^2 \ln p(\beta|\mathbf{Y})}{\partial\beta\partial\beta'}\right)^{-1}\Bigg|_{\beta=\hat{\beta}},\ \mathrm{df}\right],$$

  where $\hat{\beta}$ is obtained by R steps ($R \leqslant 3$) Newton's iterations during the proposal with analytical gradients.
- Variable selections are carried out simultaneously.
- **The key:** The analytical gradients require the derivative for the copula density and marginal densities.

The stock returns, a revisit

The tail-dependence and Kendall's τ over time (posterior mode)

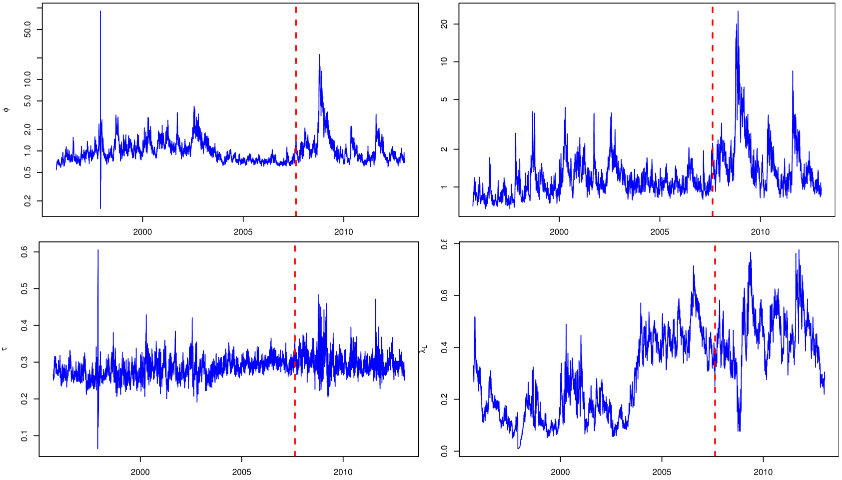


Table: Posterior summary of copula model with S&P100 and S&P600 data. In the copula component part, the first row and second row for β and $\mathcal{I}$ are the results for the combined covariates that are used in the first and second marginal model, respectively. The intercept are always included in the model.

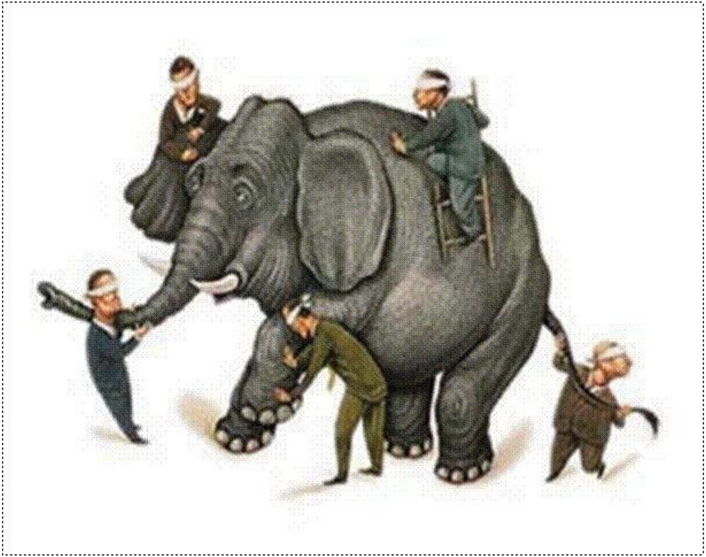
	Intercept	RM1	RM5	RM20	CloseAbs95	CloseAbs80	MaxMin95	MaxMin80	CloseSqr95	CloseSqr80
					Copula component ($\mathcal{C}$)					
β_{λ_L}	−8.165	**−0.555**	1.793	**0.005**	−0.170	**0.110**	−0.667	−1.448	−0.636	0.050
		1.463	**0.405**	0.934	−2.138	−1.288	−1.954	−1.577	**−1.873**	−1.805
$\mathcal{I}_{\lambda_L}$	1.00	**0.98**	0.37	**0.63**	0.02	**0.61**	0.36	0.35	0.39	0.29
		1.00	**1.00**	0.00	0.30	0.35	0.40	0.00	**0.61**	0.34
β_{τ}	−1.726	0.181	−0.217	−0.304	−0.107	**0.115**	**0.005**	**−0.257**	**1.068**	0.037
		−0.191	**0.170**	0.274	0.144	**−0.051**	**−0.671**	0.059	−0.209	−0.181
$\mathcal{I}_{\tau}$	1.00	0.00	0.00	0.00	0.00	**0.90**	**0.99**	**1.00**	**0.85**	0.00
		1.00	**1.00**	0.00	0.00	**1.00**	**1.00**	0.00	0.00	0.00

The inefficiency factors for the parameters are all bellow 25.

Extensions and future work

- Our bivariate tail-dependence method can be other higher-order multivariate models.
- Mixtures of copulas.
- Modeling multivariate volatility surface with copulas
- Our copula method is general and can also be applied to other areas, e.g. optimal design for multivariate data.

Can we have a model that is big like an elephant?



by John Godfrey Saxe (1816-1887)

Knowing the elephant

- Sophisticated models are essential for such situations.
- In principle, the complicated model should be able to capture more complicated data features.
- Estimating such model is not easy.
- There is huge space to explore.
  - Techniques like parallel MCMC should be explored to speed up the computation.
  - Statistics with big data is the new challenge.

Thank you!

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